Lecture 20: Dynamics of rotating systems

The basic law we know in dynamics is Newton's second law which relates force to acceleration. We also can treat dynamics using the work/energy method and we can also treat it using the fact that impulse is equal to the change in momentum. These methods always need to be used when translational motion occurs.

However when we have extended objects, we also need to consider their their rotational motion and then it is better to use angular acceleration. We have seen that for an extended object (like a beam) to be at equilibrium, we need the sum of the torques as well as the sum of the forces to be zero. What happens if the sum of the forces is zero, but the sum of the torques is not zero???

Torque and angular acceleration

If the sum of the torques is not zero then the body has an angular acceleration. This is seen by considering a mass at the end of a rod which is very stiff, but massless. Consider applying a force of magnitude F to a mass m which is attached to the end of the rod, which has length r. Newton's second law applied to the mass implies that,

$$F = ma \tag{1}$$

Where we assumed that the force is tangent to the circle. The acceleration is in a direction tangent to a circle of radius r, so the mass rotates. Now note that $a = r\alpha$, so we may write,

$$F = mr\alpha \tag{2}$$

Now multiply both sides of this equation by r to find,

$$rF = mr^2\alpha \tag{3}$$

Notice that the left hand side is torque τ, so we may write,

$$\tau = I\alpha \tag{4}$$

where $I = mr^2$. This equation gives a relation between torque and angular acceleration and is the rotational analogy to $F = ma$. The equation $\tau = I\alpha$

is very general and applies to any body, not just the simple one we considered above. The only thing that changes is the dependence of the moment of intertia, I on the total mass of the object and the radius of the object.

Moment of inertia

The moment of inertia of a body tells us how easy it is to rotate the body. The bigger the mass the larger the moment of inertia is. However the distance a mass is from the axis of rotation is also important. When talking about moment of inertia, we have to also specify the axis about which I is calculated. Here are some cases:
(i) A cylindrical shell $I = MR^2$ (about central axis)
(ii) A solid cylinder $I = MR^2/2$ (about central axis)
(iii) A solid sphere $I = 2MR^2/5$ (about axis through center)
(iv) Spherical Shell $I = 2MR^2/3$ (about axis through center)
(v) Rod $I = ML^2/12$ (about center)
(vi) Rod $I = ML^2/3$ (about one end)
If we have a set of masses we can find the moment of inertia of these masses about a rotation axis, by calculating,

$$I = \sum m_i r_i^2 \tag{5}$$

where r_i is the shortest distance from mass i to the axis of rotation. The kinetic energy stored in a rotating system is given by,

$$KE_{rot} = \frac{1}{2} I \omega^2 \tag{6}$$

The work-energy formula

$$W_e = \Delta KE + \Delta PE + E_{dissipated} \tag{7}$$

still applies, however the kinetic energy now has a sum of two terms. The translational kinetic energy $mv^2/2$ and the rotational kinetic energy $I\omega^2/2$. This is very important in the case of rolling motion, as we shall discuss below.

The angular momentum of a rotating body has magnitude,

$$L = I\omega \tag{8}$$

Note that *angular momentum is conserved if no external torque acts on a system.* Remember that linear momentum is conserved if no external force acts on a system. Examples of problems where no external torque is applied to a system are (i) When someone jumps onto a merry go round in the radial direction, (ii) When an external force reduces the orbit radius of rotational motion by applying a force in the radial direction. Why is there no torque in these cases? Is kinetic energy conserved in these processes? Is any work done by an external force in these cases?

Rolling motion (without slipping)

Rolling is a combination of rotational and translational motion. The center of the rolling object (e.g. a cylinder, a sphere or a wheel) moves along at the same height above the ground. The center of the object undergoes translational motion. At the same time, there is rotational motion. When the object rolls through one rotation, the center of object moves a distance $\delta x_c = 2\pi r$, where r is the radius of the object. This can be generalized to write,

$$\delta x_c = r\theta \tag{9}$$

Similar relations relate the center of mass velocity and acceleration to the angular velocity and acceleration, ie,

$$v_c = \omega r; \quad a_c = \alpha r \tag{10}$$

Note that this means that the velocity and acceleration of the center of a rolling object are the same as those of the edge of the object, provided we consider only the rotational motion. The kinetic energy of a rolling object moving at speed $v = v_c$ is then,

$$KE_{rol} = KE_{trans} + KE_{rot} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \tag{11}$$

Using $v = v_c = \omega r$, we find that,

$$KE_{rol} = \frac{1}{2}(MR^2 + I)\omega^2 \tag{12}$$

Example: Rolling down a hill. Consider a body that rolls (eg a sphere, cylinder etc) which is initially at rest at the top of a hill of height h. If one of these objects is released and allowed to roll without slipping, what is its angular velocity and center of mass velocity at the bottom of the hill.

Using energy conservation, we have,

$$Mgh = KE_{rol} = \frac{1}{2}(MR^2 + I)\omega^2 \tag{13}$$

Solving for ω yields,

$$\omega = (\frac{2Mgh}{I + MR^2})^{1/2} \tag{14}$$

Furthermore, we can write $I = \beta MR^2$ which leads to,

$$\omega = (\frac{2gh}{\beta R^2 + R^2})^{1/2} = \frac{(2gh)^{1/2}}{R}\frac{1}{(\beta + 1)^{1/2}} \tag{15}$$

The center of mass velocity is $v_c = \omega R$, so we find that the center of mass velocity is given by,

$$v_c = \frac{(2gh)^{1/2}}{(\beta + 1)^{1/2}} \tag{16}$$

From Eqs. (15) or (16) it is evident that if β is larger (which means larger moment of inertia), then ω and v_c are smaller. Based on this formula, we can figure out whether a solid sphere rolls down a hill faster than a solid cylinder of the same radius. Does the mass of the object matter? Why not? Does the radius of the object matter? How does the speed of these objects rolling down the hill compare to a frictionless block sliding down the same hill?

Finally note that if the angle of the hill is too large, the cylinder or sphere will slide instead of roll. The condition for the critical angle for sliding is the same as that for sliding of a block down an inclined plane, ie,

$$mgSin(\theta_c) = \mu_s N = \mu_s mgCos(\theta_c) \tag{17}$$

which gives, $Tan(\theta_c) = \mu_s$. For $\theta > \theta_c$ sliding occurs, while if $\theta < \theta_c$ then rolling occurs.